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SINGULARITIES OF THE EULER EQUATION
AND HYDRODYNAMIC STABILITY

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ABSTRACT

Equations governing the motion of a specific class of singularities of the Euler equation in the extended complex spatial domain are derived. Under some assumptions, it is shown how this motion is dictated by the smooth part of the complex velocity at a singular point in the unphysical domain. These results are used to relate the motion of complex singularities to the stability of steady solutions of the Euler equation. A sufficient condition for instability is conjectured. Several examples are presented to demonstrate the efficacy of this sufficient condition which include the class of elliptical flows and the Kelvin-Stuart Cat's Eye.

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1. INTRODUCTION

Singularities of the Euler equation for inviscid incompressible flow have been studied extensively for many years. An important question that remains quite controversial is whether, in three dimensions, an Euler flow can develop a singularity in finite time when the initial conditions are smooth (see Pumir & Siggia[1], Majda[2], Kerr[3], Brachet et al[4] and the references therein). While this paper does not address this controversy directly, except within a narrow class of singularities under restrictive assumptions, we expect that the approach presented here may help in a fuller resolution of this problem. All previous singularity studies have been limited to examining solutions to the Euler equation in the real (physical) spatial domain, though in some cases the singularity presence in the complex unphysical plane was deduced from monitoring the Fourier coefficients.

For simpler Euler flows with sharp discontinuities like the Kelvin-Helmholtz problem, previous work by Moore[5,6], Caflisch & Orlenna[7] and Cowley et al[8] have shown the advantage of studying the equations in the complex unphysical plane. In that case, the singularity formation in the physical domain can be understood in terms of two processes: (a) instantaneous singularity formation at some points in the complex unphysical domain at the initial time, and (b) the motion of such complex singularities towards the physical domain. The question of finite time singularities in the real domain then becomes a question of whether such singularities approaching the real domain actually impinge it in finite time. Moore[5,6] and Caflisch & Orlenna[7] have shown that for specific symmetric initial conditions, the singularities form on the imaginary axis after a finite time, which then move towards the origin leading to a singularity formation on the real axis. However, recent work by Cowley et al[8] suggests that the Moore singularity[5,6] actually forms at the initial instant of time at points off the imaginary axis. Later on, some of these singularities merge in pairs on the imaginary axis before traveling towards the origin (which is part of the real domain). Indeed, it appears that in a wide class of differential equations including the familiar one-dimensional inviscid Burger's equation, singularities that show up later in the real domain are either present at some complex location initially or are formed at the initial instant of time. Such singularities typically form at points (including infinity) in the complex spatial plane where a regular perturbation series in time (for early times) becomes disordered. In some cases, it is found[8,9] that singularities are born at infinity in the sense that the singularity location in the finite spatial plane recedes to infinity as $t \rightarrow 0^+$. The study of complex singularities appears to have been a fruitful exercise that transcends the question of finite time singularity. Indeed, recent work on an interfacial evolution problem by Tanveer[10] suggests that this may be an important step in understanding the effect of small regularization on an otherwise ill-posed time-evolving flow in the physical domain.

Despite successes in each of the cases cited above, and perhaps in other cases unknown to the authors, the problems were reducible to one independent space-like variable besides time. We are unaware of attempts to generalize these ideas for multiple spatial variables as appears necessary for studying a general Euler flow. Here, we study a class of moving singularities of the Euler equation whose form appears to remain unchanged with time, with the expectation that, for certain other initial conditions, the process (a) described in the last paragraph leads to the formation of complex singularities of this form when they are not present initially.

In this paper, we provide analytical evidence to show how each singularity of a specific type moves with the smooth part of the complex velocity field evaluated at that point. Of course, in a simpler vein, it has long been known that point vortex singularities in the physical domain move with the local velocity at the singularity location[11]. However, with certain assumptions, this result is generalizable to singularity motion in the complex unphysical plane as well. This result is used to conjecture sufficient conditions for the instability of a steady solution of the Euler equation. We present two conjectures. Conjecture 1 addresses the absolute instability of any steady Euler flow with a stagnation point. Conjecture 2 addresses instability in the Langrangian sense; when the conditions of this conjecture hold, there exist disturbances that grow as we follow some fluid particle.

Several applications to the hydrodynamic stability of inviscid flows in an infinite or periodic domain – which include the class of elliptical flows and the Kelvin-Stuart Cat's Eye – are made to illustrate the use of Conjecture 1 to determine the absolute instability of these flows.

2. MATHEMATICAL EQUATIONS IN THE COMPLEXIFIED DOMAIN

Consider the Euler equation for the motion of an invisicid fluid where the vorticity $\vec{\omega}(\vec{x}, t)$ and the velocity $\vec{u}(\vec{x}, t)$ are determined by

$$\vec{\omega}_t + \vec{u} \cdot \nabla\vec{\omega} - \vec{\omega} \cdot \nabla\vec{u} = 0 \tag{1}$$

$$\nabla \cdot \vec{u} = 0 \tag{2}$$

$$\nabla \times \vec{u} = \vec{\omega} \tag{3}$$

Each of the components of the above equations can be put in the form

$$F(x_1, x_2, x_3, t) = 0 \tag{4}$$

where $\vec{x} = (x_1, x_2, x_3)$. In the case of 2π periodicity in each direction, if we assume that each coefficient C_{jkl} in the following representation of F

$$F(x_1, x_2, x_3) = \sum_{j,k,l} C_{jkl}\, e^{i[jx_1 + kx_2 + lx_3]} \tag{5}$$

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satisfies the relation

$$|C_{jkl}| \; < \; K \; e^{-\delta(|j|+|k|+|l|)} \tag{6}$$

for some constant K and δ which are independent of j, k and l, then it follows that (4) will hold for complex x_1, x_2 and x_3 when $|Im \; x_1|$, $|Im \; x_2|$ and $|Im \; x_3|$ are each small. Since the right hand side of (4) is zero, which is an analytic function of x_1, x_2 and x_3 in the complex domain, it follows that (4) holds for all complex $\vec{x}$. The condition (6) will certainly hold for the Fourier series coefficient of each component of equation (1), (2) and (3) if each component of $\vec{u}$ and $\vec{\omega}$ and its derivatives satisfy such a condition. We restrict our attention to cases when this is true at some early time, which we assume is possible for a certain subclass of initial conditions. Then we arrive at the conclusion that each of equations (1), (2) and (3) hold for complex $\vec{x}$ (i.e. each component is complex). However, since the right hand side of each component of (1), (2) and (3) is zero, which is analytic for all complex $\vec{x}$ and t, it follows that (1), (2) and (3) will hold for all $\vec{x}$ and t. The singularities of $\vec{u}$ and $\vec{\omega}$ are those that are consistent with the solutions of (1), (2) and (3) in the complex $\vec{x}$ domain. The same arguments hold even when the periods in x_1 , x_2 and x_3 are not the same or when the geometry is infinite in some directions and periodic in others.

We examine a certain kind of initial conditions for which the velocity and vorticity are each real and analytic everywhere for real $\vec{x}$, but have complex singularities at a singularity surface determined by

$$d(\vec{x}, 0) = 0 \tag{7}$$

where d is an analytic function of each component of $\vec{x}$ that is real valued and greater than 0 for real $\vec{x}$. Here d is also a function of t and its time evolution equation will be derived later. We also require the non-degeneracy condition that $\nabla d \; \neq \; 0$ where $d = 0$[‡]. We consider initial conditions that can be decomposed into

$$\vec{u}(\vec{x}, 0) = \vec{u}_s(\vec{x}, 0) + f(d(\vec{x}, 0)) \; \vec{q}(\vec{x}, 0) \tag{8}$$

$$\vec{\omega}(\vec{x}, 0) = \vec{\omega}_s(\vec{x}, 0) + f'(d(\vec{x}, 0)) \; \vec{p}(\vec{x}, 0) \tag{9}$$

where

$$f(d) = d^\alpha \tag{10}$$

with $0 \; < \; \alpha \; < \; 1$. Thus the initial vorticity blows up at the complex singular points, though the velocity is finite. In (8) and (9), we assume that each of $\vec{u}_s(\vec{x}, 0)$, $\vec{\omega}_s(\vec{x}, 0)$, $\vec{q}(\vec{x}, 0)$ and $\vec{p}(\vec{x}, 0)$ are analytic functions of $\vec{x}$ and that they are real valued for real $\vec{x}$. Thus the initial

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‡We have shown later that this condition is satisfied at later times once satisfied initially.

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conditions (8) and (9) correspond to smooth initial conditions in the real domain. We also require that

$$\nabla \cdot \vec{u}_s(\vec{x}, 0) = 0 \tag{11}$$

$$\nabla \times \vec{u}_s(\vec{x}, 0) = \vec{\omega}_s(\vec{x}, 0) \tag{12}$$

and

$$\nabla \times [f \, \vec{q}(\vec{x}, 0)] = f' \vec{p}(\vec{x}, 0) \tag{13}$$

Now, for $t > 0$, we seek solutions to (1) - (3) of the form

$$\vec{u}(\vec{x}, t) = \vec{u}_s(\vec{x}, t) + f(d(\vec{x}, t)) \, \vec{q}(\vec{x}, t) \tag{14}$$

$$\vec{\omega}(\vec{x}, t) = \vec{\omega}_s(\vec{x}, t) + f'(d(\vec{x}, t)) \, \vec{p}(\vec{x}, t) \tag{15}$$

From (2) and (3), we get the following relations:

$$\nabla \cdot \vec{u}_s + f' \left[\frac{f}{f'} \nabla \cdot \vec{q} + \vec{q} \cdot \nabla d \right] = 0 \tag{16}$$

$$\nabla \times \vec{u}_s + f' \left[\frac{f}{f'} \nabla \times \vec{q} + \nabla d \times \vec{q} \right] = \vec{\omega}_s + f' \vec{p} \tag{17}$$

If we now require that

$$\nabla \cdot \vec{u}_s = 0 \tag{18}$$

and

$$\nabla \times \vec{u}_s = \vec{\omega}_s \tag{19}$$

$$\vec{\omega}_{s_t} + \vec{u}_s \cdot \nabla \vec{\omega}_s - \vec{\omega}_s \cdot \nabla \vec{u}_s = 0 \tag{20}$$

then the relations (16) and (17) imply that

$$\frac{f}{f'} \nabla \cdot \vec{q} + \vec{q} \cdot \nabla d = 0 \tag{21}$$

$$\frac{f}{f'} \nabla \times \vec{q} + \nabla d \times \vec{q} = \vec{p} \tag{22}$$

In addition, the substitution of (14) and (15) into (1), with the use of relations (20) - (22), yields

$$f'' \vec{p} [d_t + \vec{u}_s \cdot \nabla d] + f' \left[\vec{p}_t + \vec{u}_s \cdot \nabla \vec{p} - \vec{p} \frac{f^2 f''}{f'^2} \nabla \cdot \vec{q} + f(\vec{q} \cdot \nabla) \, \vec{p} + \frac{f}{f'}(\vec{q} \cdot \nabla) \, \vec{\omega}_s \right.$$

$$\left. - \vec{q}(\vec{\omega}_s \cdot \nabla \, d) - (\vec{p} \cdot \nabla) \, \vec{u}_s - \frac{f}{f'} (\vec{\omega}_s \cdot \nabla) \, \vec{q} - \vec{q} f (\nabla d) \cdot (\nabla \times \vec{q}) - f \vec{p} \cdot \nabla \vec{q} \right] = 0 \tag{23}$$

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Now notice that if we choose to evolve $d(\vec{x}, t)$ according to

$$d_t + \vec{u}_s \cdot \nabla d = 0 \qquad (24)$$

then

$$\vec{p}_t + \vec{u}_s \cdot \nabla \vec{p} - p\frac{f^2 f''}{f'^2} \nabla \cdot \vec{q} + f(\vec{q} \cdot \nabla)\,\vec{p} + \frac{f}{f'}(\vec{q} \cdot \nabla)\,\vec{\omega}_s = \vec{q}(\vec{\omega}_s \cdot \nabla d)$$

$$+ (\vec{p} \cdot \nabla)\,\vec{u}_s + \frac{f}{f'}(\vec{\omega}_s \cdot \nabla)\,\vec{q} + \vec{q}f(\nabla d) \cdot (\nabla \times \vec{q}) + f\vec{p} \cdot \nabla \vec{q} \qquad (25)$$

The existence of $\vec{p}$ satisfying (21), (22) and (25) for small enough time follows from the existence of $\vec{\omega}$ and $\vec{\omega}_s$, each satisfying (1) - (3), and of d satisfying (24), since $\vec{\omega} - \vec{\omega}_s = f'(d)\vec{p}$. We also notice that the coefficients in (21),(22) and (25) are continuous even at $d = 0$. Therefore, it appears reasonable to assume that for small enough time, with appropriately regular initial data, such a solution is C^1 in x_1, x_2 and x_3 at $d = 0$ in the extended complex domain. Since, (24) implies that $d > 0$ in the real domain, once it is so initially, it follows that such a solution $\vec{p}$ will be real and analytic in the real domain for sufficiently small t since it is so initially. Notice that if we were to choose $d_t + \vec{u}_s \cdot \nabla d \neq 0$ at $d = 0$, then the resulting evolution equation for $\vec{p}$ would contain singular coefficients in $\vec{x}$ at the singularity surface $d = 0$, which would preclude a C^1 solution for $\vec{p}$ at $d = 0$. Indeed for a general choice, $\vec{p}$ is not even expected to be finite at $d = 0$.

Despite the choice (24) for the evolution of d – which results in (25) having C^1 coefficients when $d = 0$ – we cannot rule out the possibility that the system of nonlinear evolution equations for $\vec{\omega}_s$, $\vec{p}$ and d develops spontaneous singularities after a finite time. Hence, global existence in time cannot be expected in general. However, we assume that there exists a nontrivial subclass of initial conditions so that no spontaneous singularity develops in the complex domain for $\vec{p}$, $\vec{q}$, $\vec{u}_s$ or $\vec{\omega}_s$, other than a weak singularity of $\vec{p}$ and $\vec{q}$ at $d = 0$ (in the sense that $\vec{p}$ is C^1 but not analytic). In that case, the continuity of $\vec{p}$ at $d = 0$ implies that at later times, the singularity of $\vec{\omega}$ is still determined by

$$d(\vec{x}, t) = 0 \qquad (26)$$

where f' is singular. Writing (24) in the characteristic form, it is clear that (26) will be satisfied for points $\vec{x}_0(t)$ determined by

$$\frac{d\vec{x}_0(t)}{dt} = \vec{u}_s(\vec{x}_0(t), t) \qquad (27)$$

with the initial condition $\vec{x}_0(0)$ satisfying the constraint that

$$d(\vec{x}_0(0), 0) = 0 \qquad (28)$$

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Thus, each point on the singularity surface moves with the speed $\vec{u}_s$ evaluated at that point. However, $\vec{u}_s$ is the smooth part of the vector field $\vec{u}$ at $d = 0$ in the complex $\vec{x}$ plane, which means the location of complex singularities of the type considered here can be determined without having to calculate $\vec{p}(\vec{x}, t)$ from (25). Furthermore, with the assumption on the smoothness of $\vec{u}_s$ and continuity of $\vec{p}$ for all time, any singularity of this kind can only approach the physical domain without actually impinging it in finite time. This can be deduced from (24) by putting it in the characteristic form and noting that $\vec{u}_s$ is real in the real domain and $d(\vec{x}, 0) > 0$ in the real physical domain. Furthermore, if a flow has a real point (say the origin, without loss of generality) where $\vec{u}_s = 0$ with a velocity gradient tensor $T_{kj} = \frac{\partial u_{s_k}}{\partial x_j}$ independent of time, then replacing $\vec{u}_s$ in (27) by $\mathbf{T}\,\vec{x}$ shows that a complex singularity can only approach the physical domain exponentially in time. This observation may be relevant to the numerical calculation of Brachet et al[4] in the Taylor-Green problem, where they note an exponential approach of a singularity towards the physical domain. However, since the deformation tensor is traceless, because of the divergence condition, it is clear that only a restricted set of points on the initial singularity surface are a subset of the complex stable manifold of the fixed point[§].

Equation (27) is convenient since it shows that one need not solve for $\vec{p}(\vec{x}, t)$ to determine the singularity location in the complex plane even though one needs to determine $\vec{u}_s(\vec{x}, t)$ in the complex plane. When exact solutions are known for $\vec{u}_s$, which may be steady or unsteady, this poses no problems; however, when the solution is only determined numerically in the real domain, one cannot expect to determine $\vec{u}_s$ in the complex plane through Fourier extrapolation, since such a procedure is ill-posed. In such a case, the solutions have to be determined numerically, directly in the complex plane. This will be the subject of another paper. However, if the singularities are close to the physical domain, it is possible to obtain complex singularity trajectories directly from $\vec{u}_s$ in the real domain. This can be seen by decomposing $\vec{x}_0(t) = \vec{x}(t) + i\vec{y}(t)$, into its real and imaginary parts and taking the real and imaginary part of (27) in the limit of small $\vec{y}$. We then get

$$\frac{d\vec{x}}{dt} = \vec{u}_s(\vec{x}(t), t) \tag{29}$$

$$\frac{d\vec{y}}{dt} = \nabla \vec{u}_s(\vec{x}(t), t)\ \vec{y} \tag{30}$$

Equations (29) and (30) are the approximate singularity trajectory equations when a complex singularity is close to the physical domain. In this case, one needs $\vec{u}_s$ and its gradient in only the real domain, which can be determined numerically.

[§]Complex stable manifold of a real fixed point of (27) is defined as the set of complex initial conditions $\vec{x}_0(0)$ such that $\vec{x}_0(t)$, satisfying (27), tends to the fixed point as $t \to \infty$.

Once the singularity locations are determined, one can determine $\vec{p}(\vec{x}_0(t), t)$ and $\nabla d(\vec{x}_0(t), t)$ in terms of a relatively simple system of ordinary differential equations, as will be derived shortly. These quantities are of interest since for a point $\vec{x}$ sufficiently close to $\vec{x}_0(t)$ (so that d is small) but not on the singularity surface, we have

$$\vec{\omega} \sim \alpha[\nabla d(\vec{x}_0(t), t) \cdot (\vec{x} - \vec{x}_0(t))]^{-(1-\alpha)} \vec{p}(\vec{x}_0(t), t) \tag{31}$$

For brevity of notation, we define

$$\vec{A}(t) = \nabla d(\vec{x}_0(t), t) \tag{32}$$

$$\vec{B}(t) = \vec{q}(\vec{x}_0(t), t) \tag{33}$$

$$\vec{C}(t) = \vec{p}(\vec{x}_0(t), t) \tag{34}$$

Then, it is clear that, since $d(\vec{x}_0(t), t) = 0$ and each of f, f/f' are zero at $d = 0$, then (21) and (22) implies that

$$\vec{A} \cdot \vec{B} = 0 \tag{35}$$

$$\vec{A} \times \vec{B} = \vec{C} \tag{36}$$

It follows that

$$\vec{C} \times \vec{A} = (\vec{A} \cdot \vec{A})\vec{B} \tag{37}$$

Furthermore, it is clear that (25) implies that

$$\frac{d\vec{C}}{dt} = (\vec{\omega}_s \cdot \vec{A})\vec{B} + \mathbf{T}\vec{C} \tag{38}$$

where $\mathbf{T}$ is a second rank tensor whose elements are defined by

$$T_{jk} = \frac{\partial u_{sj}}{\partial x_k} \tag{39}$$

Assuming $\vec{A} \cdot \vec{A} \neq 0$[¶] we get from (37) that

$$\frac{d\vec{C}}{dt} = \mathbf{S}\vec{C} \tag{40}$$

where $\mathbf{S}$ is a second rank tensor whose elements are given by

$$S_{jk} = T_{jk} + \epsilon_{jkl} A_l \frac{\vec{\omega}_s \cdot \vec{A}}{\vec{A} \cdot \vec{A}} \tag{41}$$

[¶]Note that it is possible for $\vec{A} \cdot \vec{A} = 0$ without $\vec{A}$ being zero since the dot product defined here involves no complex conjugate. However, this is not so if $\vec{A}$ is real or purely imaginary.

given that ϵ_{jkl} is the usual Levi-Civita tensor and the Einstein summation convention on repeated indices has been used.

We now argue that $\vec{A}$ remains nonzero when initially nonzero. By taking the gradient of (24), it is clear that

$$\frac{d\vec{A}}{dt} = -\mathbf{T}^{+}\vec{A} \tag{42}$$

where the $^{+}$ superscript refers to the transpose of $\mathbf{T}$ defined by (39). Therefore, the only way for $\vec{A}$ to be zero in finite time is if the elements of the tensor $\mathbf{T}$ blow up in finite time. If $\vec{u}_s(\vec{x}, t)$ is smooth for all times and grows at infinity (for an unbounded domain) no faster than linearly in $\vec{x}$, then clearly $\mathbf{T}(\vec{x}_0(t), t)$ cannot blow up in finite time. Thus, $\vec{A}$ cannot be zero in finite time when initially nonzero. Equations (40) and (42) form a closed system of equations for $\vec{A}$ and $\vec{C}$, and thus for $\nabla d(\vec{x}_0(t), t)$ and $\vec{p}(\vec{x}_0(t), t)$. This system of ordinary differential equations can be solved numerically once $\vec{u}_s$ and its spatial derivatives are known in the complex plane. In the limiting case of a singularity approaching the physical domain, further simplifications are possible making it feasible to calculate $\vec{A}$ and $\vec{C}$ in an approximate manner from knowledge of $\vec{u}_s$ and its gradient in the real domain only.

The motion of complex singularities of vorticity is significant since its approach towards the physical domain can be associated with the appearance of small scale structures at places in the real domain that are closest to the complex singularities. Furthermore, it is clear that the location of initial singularities in the complex plane is related to the data specification in the real domain in an ill-posed way; i.e., one can specify data in the real domain with arbitrary but finite precision, yet one cannot nail down the singularity location in the complex spatial domain to any precision. Yet, if the complex singularities approach the physical domain along some trajectories, one can explain the sensitivity of the later time evolution in the physical domain to initial conditions. One can conceivably start with a random distribution of singularities in the complex domain and generate an understanding of where and when in a physical flow one can expect fine scale structures where viscosity is important. The full time-dependent calculation will be the subject of a future investigation.

3. POSSIBLE IMPLICATIONS FOR THE HYDRODYNAMIC STABILITY OF STEADY FLOWS

In order for a flow to be stable, any disturbance to a basic steady flow cannot grow without bound as $t \to \infty$. A flow will be considered absolutely unstable here if there is a continual growth of disturbance vorticity at some fixed Eulerian point in the flow field. It will be considered unstable in a Langrangian sense if there exists a disturbance that grows at some real point that moves with the steady flow.

If we consider a basic steady flow, and associate the steady flow velocity and vorticity with

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$\vec{u}_s(\vec{x})$ and $\vec{\omega}_s(\vec{x})$, respectively, as described in the last section, we can consider superposition of an initial disturbance so that the initial condition is of the form (8) and (9). Our arguments in the last section suggest that any point $\vec{x}_0(t)$ on the singularity surface $d(\vec{x}_0(t), t) = 0$ will move according to (27). Since $\vec{u}_s$ is now a steady flow, (27) reduces to

$$\frac{d\vec{x}_0}{dt} = \vec{u}_s(\vec{x}_0(t)) \qquad (43)$$

When a singularity continually moves closer towards the physical domain, one can expect smaller and smaller scales to be amplified in the physical domain at some physical point in the immediate vicinity of the complex singularity. More precisely, in the limit of a complex singularity approaching the real domain, its distance from the real domain will be of the same order as the small scales generated in the real domain. As discussed before, singularities of the form considered here can at best approach the physical domain continually without actually impinging it in finite time.

The relevant question, as far as hydrodynamic stability in the absolute sense is concerned, is whether the disturbance vorticity $f'(d(\vec{x}, t))\vec{p}(\vec{x}, t)$ grows indefinitely with t at some real point $\vec{x}$. Since d can never be zero in the real domain and must be bounded below by the smallest value of d on the real domain at the initial time (which follows from (24)), it is clear that the growth of disturbance vorticity will depend on the growth of $\vec{p}$ in the real domain.

We will now show that if there is a trajectory $\vec{x}_0(t)$ determined from (43) that asymptotes to some real fixed $\vec{x}_s$ as $t \rightarrow \infty$, then it is possible to choose a disturbance such that $\vec{p}(\vec{x}_0(t), t)$ grows exponentially in time.

Without any loss of generality, we will choose $\vec{x}_s = 0$. Then, according to the given condition, there exists an $\vec{x}_0(t)$ such that it approaches the origin as $t \rightarrow \infty$. Near the origin, i.e. the stagnation point, it is clear that

$$\vec{u}_s(\vec{x}) \sim \mathbf{T}\vec{x} \qquad (44)$$

where $\mathbf{T}$ is now a constant matrix. Then according to (43) and (44), for $t > t_0$ (where t_0 is sufficiently large), $\vec{x}_0(t)$ is determined approximately by

$$\frac{d\vec{x}_0}{dt} = \mathbf{T}\,\vec{x}_0 \qquad (45)$$

According to our condition of approach to the origin, $\mathbf{T}$ must have at least one eigenvalue σ with a negative real part. We choose $\vec{x}_0(t_0)$ to be in the eigenspace corresponding to σ. If σ is real, $\vec{x}_0(t_0)$ will be chosen to be completely imaginary. Whatever the case, we have

$$\vec{x}_0(t) = \vec{x}_0(t_0)e^{\sigma(t-t_0)} \qquad (46)$$

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Then, if we choose $\vec{A}(t_0) = \vec{\xi}$ to be in the eigenspace of $\mathbf{T}^+$ corresponding to the eigenvalue σ, according to (42), it follows that

$$\vec{A}(t) = \vec{\xi} e^{-\sigma(t-t_0)} \tag{47}$$

From (46) and (47), it is clear that $\vec{A} \cdot \vec{x}_0$ is bounded as $t \to \infty$. Equation (40) now holds where the expression (41) for S_{jk} is given by

$$S_{jk} = T_{jk} + \epsilon_{jkl}\xi_l \frac{\vec{\omega}_s \cdot \vec{\xi}}{\vec{\xi} \cdot \vec{\xi}} \tag{48}$$

Clearly, since $\vec{\xi}$ is a left eigenvector of $\mathbf{T}$ (i.e. an eigenvector of $\mathbf{T}^+$), it is clear from (48) that it is also a left eigenvector of $\mathbf{S}$. Thus the set of eigenvalues of $\mathbf{T}$ and $\mathbf{S}$ are identical. Since the velocity gradient tensor $\mathbf{T}$ is traceless (incompressibility constraint), it is clear that the sum of eigenvalues of $\mathbf{T}$, and therefore of $\mathbf{S}$, must be 0. Thus, since there is an eigenvalue σ with a strictly negative real part, there must exist some other eigenvalue λ with a positive real part. If $\vec{\zeta}$ is a nonzero vector of $\mathbf{S}$ in the subspace corresponding to eigenvalue λ, it follows that if $\vec{C}(t_0) = \vec{\zeta}$, then

$$\vec{C}(t) = \vec{\zeta} e^{\lambda(t-t_0)} \tag{49}$$

Thus, clearly there exist disturbances for which $\vec{C}$ and therefore $\vec{p}(\vec{x}_0(t), t)$ grows exponentially as $t \to \infty$. Note that the choice made in (49) is perfectly consistent with the condition $\vec{C} \cdot \vec{A} = 0$ that follows from (37), since $\vec{\xi}$ and $\vec{\zeta}$ are left and right eigenvectors of $\mathbf{S}$ corresponding to differing eigenvalues σ and λ. Our proof that $\vec{p}(\vec{x}_0(t), t)$ grows exponentially in t when $\vec{x}_0(t)$ approaches $\vec{x}_s$ requires that $\vec{\xi} \cdot \vec{\xi} \neq 0$. In the unusual case when there is no eigenvector of $\mathbf{T}^+$ with this property, a modified analysis is required by choosing a purely imaginary vector $\vec{\xi}$ in the subspace corresponding to the eigenvalues σ and σ^* (Note that eigenvalues with a negative real part must be complex in this case). Then, the tensor elements S_{jk} in (48) will not be constants but are periodic in time. In that case, we have been unable to show in any elementary way that $\vec{C}(t)$ must grow with time, though we expect this to be the case.

Note, however, that for two-dimensional disturbances superposed on a two-dimensional flow with a stagnation point, $\vec{C}(t_0)$ must necessarily be aligned along the x_3 axis, which is an eigenvector of $\mathbf{T}$ corresponding to eigenvalue 0; therefore, it cannot be chosen to be $\vec{\zeta}$. Indeed, for two dimensional disturbances, the terms on the right hand side of (38), which arise from vortex stretching, are absent and $\vec{C}$ is a constant.

From (24), it is clear that at the stagnation point $\vec{r}_s$, $d(\vec{x}_s, t) = d(\vec{x}_s, 0)$. Thus if $d(\vec{x}_s, 0)$ is chosen sufficiently small, each of f, f/f', $f^2 f''/f'^2$ appearing in (25) is going to be uniformly small. Thus, it seems plausible that as $\vec{x}_0(t)$ approaches $\vec{x}_s$ and $\vec{p}(\vec{x}_0(t), t)$ grows

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exponentially in time, $\vec{p}(\vec{x}_s, t)$ (and hence the disturbance vorticity $f'(d)\vec{p}(\vec{x}_s, t)$) will grow. However, the mathematical proof of this statement is far from obvious for a general flow, where $\vec{p}$ is determined by the nonlinear equation (25). However, in the Appendix, under some additional assumptions which appear reasonable, we are able to show that exponential growth of $\vec{p}(\vec{x}_0(t), t)$ does correspond to a growth of disturbance vorticity $\vec{\omega}_D = f'(d)\vec{p}$ at $\vec{x}_s$ for a steady flow where $\nabla \vec{u}_s$ is a constant and $\vec{\omega}_D$ is small enough to permit linearization. We expect that the connection actually transcends these restrictions, though we have not succeeded in proving this.

Based on the arguments in this section and the results in the Appendix, we present the following conjectures.

Conjecture 1

A sufficient condition for the absolute instability of a smooth steady Euler flow in a periodic or unbounded domain is that a singularity moving in the complex plane according to (27) has the property that as $t \rightarrow \infty$, $\vec{x}_0(t) \rightarrow \vec{x}_s$ for a real $\vec{x}_s$, where $Im\ \vec{x}_0(0) \neq 0$. This condition is equivalent to the existence of a stagnation point $\vec{x}_s$, where the velocity gradient tensor $\mathbf{T}$ has at least one eigenvalue with a negative real part. In that case, the vorticity at $\vec{x}_s$ will grow in time.

Remark 1

The above conjecture is only a sufficient condition for instability, though not necessary, as shown in a later example. This is because only specific forms of disturbances have been chosen– namely, those associated with a specific singularity structure in the complex plane.

Remark 2

There is no restriction on the size of the disturbance, though there is a restriction on its form. Nonetheless, we do not expect this to be useful in uncovering nonlinear instability of flows which are linearly stable since the condition for approach of singularities is equally valid for linearized equations.

Remark 3

Arbitrary disturbances in the conjecture refer to any arbitrary disturbance in three dimensions, although the steady flow can be two-dimensional. For two-dimensional disturbances superposed on two-dimensional flow, the approach of a complex singularity can be associated with the growth of the gradient of vorticity only. This can be seen by just taking the gradient of the scalar vorticity equation and evaluating it at the stagnation point, which leads to $d\vec{C}/dt = \mathbf{T}\vec{C}$, where $\vec{C} = \nabla\omega(\vec{x}_s, t)$, and $\mathbf{T}$ is the two-dimensional velocity gradient tensor at the stagnation point.

Remark 4

This conjecture can be applied to any steady state flow – even ones that are numerically

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determined – once the deformation tensor $\mathbf{T}$ at a stagnation point is obtained.

Conjecture 2

A steady flow is unstable in the Langrangian sense if there exists a choice of initial conditions $\vec{x}(0)$, $\vec{y}(0)$, $\vec{A}(0)$ and $\vec{C}(0)$ such that the solution to (29), (30), (40) and (42) have the feature that as $t \to \infty$, $\vec{y} \to 0^{\|}$ and $\vec{C}(t) \to \infty$.

In the following subsections, we present examples of the application of Conjecture 1 to specific simple Euler flows. Conjecture 2 appears to be harder to apply; at this point, it is not clear to us how this can be applied in a fruitful manner.

3.1 Elliptical Flows

We first consider the case of plane strain in an infinite flow domain with the velocity field

$$\vec{u}_s(\vec{x}, 0) = (\alpha x_1, -\alpha x_2, 0) \tag{50}$$

with $\alpha > 0$. Then according to (27), a singularity at $\vec{x}_0 = (x_1, x_2, x_3)$ moves according to

$$\frac{dx_1}{dt} = \alpha x_1 \tag{51}$$

$$\frac{dx_2}{dt} = -\alpha x_2 \tag{52}$$

$$\frac{dx_3}{dt} = 0 \tag{53}$$

Hence

$$x_1 = x_1(0)e^{\alpha t} \quad , \quad x_2 = x_2(0)e^{-\alpha t}, \quad x_3 = x_3(0) \tag{54}$$

from which it is clear that if we choose $Re\ x_j(0) = 0$ for $1 \le j \le 3$ and $Im\ x_1(0) = Im\ x_3(0) = 0$, then $\vec{x}_0(t) \to 0$ as $t \to \infty$. According to Conjecture 1, the flow is going to be unstable in the absolute sense. This is in accordance with established results from hydrodynamic stability theory (see Pearson[12], Lagnado et al[13] and Bayley[14]).

The relation of the approach of a complex singularity to the appearance of increasingly small scale structures is highlighted by the following perturbation of our straining flow:

$$\vec{u}(\vec{x}, 0) = (\alpha x_1 + f(x_2), -\alpha x_2, 0) \tag{55}$$

for an arbitrary nontrivial differentiable function f. In this case, there is a simple exact solution[**] given by

$$\vec{u}(\vec{x}, t) = [\alpha x_1 + e^{-\alpha t}f(x_2 e^{\alpha t}), -\alpha x_2, 0] \tag{56}$$

[‖] Recall $\vec{x}_0(t) = \vec{x}(t) + i\vec{y}(t)$

[**] This is probably well known; however, we could not find a specific reference

with the corresponding two-dimensional vorticity

$$\omega = -f'(x_2 e^{\alpha t}) \tag{57}$$

Note that if the function f has a singularity in the complex plane, it approaches the physical domain as $t \rightarrow \infty$. For instance if

$$f(x_2) = \frac{1}{\sqrt{1 + x_2^2}} \tag{58}$$

then from (55) and (56), it follows that the singularity of ω and $\vec{u}$ is located at $x_2 = i\, e^{-\alpha t}$ which approaches zero as $t \rightarrow \infty$. In this case, with a two dimensional disturbance superposed on a two dimensional flow, the approach of a complex singularity does not correspond to growth of vorticity; however, as expected from *Remark 3*, the gradient of vorticity grows exponentially with time. The solution form (57) in the physical domain clearly corresponds to transfer of energy to smaller scales near $x_2 = 0$, though the energy contained at any specific scale eventually decreases with time. The similarity variable $e^{\alpha t}x_2$ in (56) could have been predicted by looking at the solution (54) in the complex plane. Note that $Im\ x_1$ and $Im\ x_2$ need not go to zero for all initial conditions in order for Conjecture 1 to hold. Indeed, there exist *initial conditions when singularities move further away from the real domain* which corresponds to a reverse cascade: *energy transfer from smaller scales to larger scales*. This can also be seen in terms of exact solutions to the Euler equation of the form (56) when the initial perturbation velocity involves the second component rather than the first.

Three-dimensional generalizations of the above straining flow can be made for which the velocity is given by

$$\vec{u}_s = (\alpha x_1, \beta x_2, \gamma x_3) \tag{59}$$

where $\alpha + \beta + \gamma = 0$ due to (11). The case of $\beta = \gamma = -\frac{1}{2}\alpha$ corresponds to the axisymmetric expansion/contraction which has been studied extensively in the turbulence literature. As $\gamma \rightarrow 0$, the plane strain case considered earlier is recovered. Here, we show how to apply Conjecture 1 in its simpler form. The velocity gradient tensor corresponding to (59) is

$$\mathbf{T} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \tag{60}$$

The eigenvalues of (60) are α, β and γ – at least one of which is negative due to the incompressibility constraint $\alpha + \beta + \gamma = 0$. Hence, according to Conjecture 1, the flow is unstable in accordance with results from hydrodynamic stability (see Pearson[12] and Townsend[15]). We will apply Conjecture 1 to the remaining elliptical flows to be considered with this simple eigenvalue test.

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Plane strain plus rotation in an infinite flow domain corresponds to the base flow

$$\vec{u}_s = (\alpha x_1 - \Omega x_2, \Omega x_1 - \alpha x_2, 0) \tag{61}$$

This physically corresponds to a plane strain (with strain rate α) combined with a solid body rotation (with angular velocity Ω) in the plaine of the strain. The velocity gradient tensor corresponding to this flow is

$$\mathbf{T} = \begin{pmatrix} \alpha & -\Omega & 0 \\ \Omega & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{62}$$

The eigenvalues of (62) are 0, $\sqrt{\alpha^2 - \Omega^2}$ and $-\sqrt{\alpha^2 - \Omega^2}$ and, hence, one of them has a negative real part if $\alpha^2 > \Omega^2$. Consequently, according to Conjecture 1, this flow will be absolutely unstable when

$$\alpha^2 - \Omega^2 > 0 \tag{63}$$

This result is consistent with recent stability analysis of elliptical flows (see Bayley[14] and Craik and Criminale[16]). Since Conjecture 1 only provides a sufficient condition for instability, it does not formally apply when $\Omega^2 > \alpha^2$. Nonetheless, it is interesting to note that $\Omega^2 > \alpha^2$ is actually an established condition for the stability of this flow (see Bayley[14]).

We now consider the base flow

$$\vec{u}_s = (Sx_2 - 2\Omega x_2, 2\Omega x_1, 0) \tag{64}$$

which serves as a model for uniform shear flow subject to a system rotation with angular velocity Ω (see Batchelor[11]). Uniform shear flow is recovered in the limit $\Omega \rightarrow 0$. The velocity gradient tensor $\mathbf{T}$ here is given by

$$\mathbf{T} = \begin{pmatrix} 0 & S - 2\Omega & 0 \\ 2\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{65}$$

The eigenvalues of (65) are 0, $\sqrt{2\Omega(S - 2\Omega)}$ and $-\sqrt{2\Omega(S - 2\Omega)}$. Hence, according to Conjecture 1, this flow is unstable if

$$2\Omega(S - 2\Omega) > 0 \tag{66}$$

since this condition guarantess the existence of a real negative eigenvalue. In terms of the rotational Richardson number $Ri = -2\frac{\Omega}{S}\left(1 - 2\frac{\Omega}{S}\right)$ first introduced by Bradshaw[17], it is clear that according to Conjecture 1, the flow is unstable if

$$Ri < 0 \tag{67}$$

This result is identical to that obtained previously by more complicated linear stability analyses (see Lezius[18] and Lezius and Johnston[19]).

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For a uniform shear flow, $Ri = 0$ and, therefore, Conjecture 1 is not useful in determining the stability or instability of the flow. The same is true for the more general case of plane shear flows where

$$\vec{u}_s = (f(x_2), 0, 0) \tag{68}$$

Of course, it is well known that uniform shear flow is weakly unstable; the kinetic energy of infinitesimal disturbance grow linearly with time[13,15]. The necessary and sufficient condition for a general plane shear flow (68) to be unstable is for $f'(x_2)$ to change sign – the classical inflexion point theorem of Rayleigh.

3.2 Kelvin-Stuart Cat's Eye

This class of exact steady solutions to the Euler equation has been presented by Stuart[20]. Without any loss of generality, his solution for the stream function Ψ can be written as

$$\Psi(x_1, x_2) = \frac{1}{2} \ln \left[\cosh (2\, x_2) + \rho \cos (2\, x_1) \right] \tag{69}$$

where $0 < \rho < 1$ is a constant characterizing the family of solutions. The velocity field $\vec{u}_s$ in this case is given by

$$\vec{u}_s = \left(\frac{\sinh 2x_2}{\cosh 2x_2 + \rho \cos 2x_1}, \; \frac{\rho \sin 2x_1}{\cosh 2x_2 + \rho \cos 2x_1}, \; 0 \right) \tag{70}$$

Within a period π in x_1, there are two distinct stagnation points at $(0,\,0)$ and $(\frac{\pi}{2},\,0)$. At $(\frac{\pi}{2}, 0)$ the velocity gradient tensor $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{pmatrix} 0 & \frac{2}{1-\rho} & 0 \\ -\frac{2\rho}{1-\rho} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{71}$$

The eigenvalues of this tensor are 0, $2i\rho^{1/2}/(1-\rho)$ and $-2i\rho^{1/2}/(1-\rho)$, none of which have a negative real part. Conjecture 1 is therefore not applicable and nothing can be deduced about the stability from the existence of this stagnation point. However, at the stagnation point $(0,\,0)$, the velocity gradient tensor is

$$\mathbf{T} = \begin{pmatrix} 0 & \frac{2}{1+\rho} & 0 \\ \frac{2\rho}{1+\rho} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{72}$$

The eigenvalues of this tensor are 0, $-2\rho^{1/2}/(1+\rho)$ and $2\rho^{1/2}/(1+\rho)$, one of which is real and negative. Hence, according to Conjecture 1, the Kelvin-Stuart Cat's Eye is unstable. This agrees with the findings of Pierrehumbert and Widnall[21].

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4. CONCLUDING REMARKS

We have considered the motion of a specific type of singularities of the incompressible three dimensional Euler equation in the complex spatial domain that correspond to a smooth variation in the real domain. Under the assumption that there is a class of initial conditions where no new singularities form in the real or complex domain other than what is present initially, we have derived an equation (namely, Eq. (27)) for the trajectory of a singularity and have shown that such a singularity cannot impinge the physical domain in finite time. Nonetheless, we expect that the approach of these singularities towards the physical domain will correspond to the formation of small scale structures therein, where the neglected viscous effects would be important. The sensitivity of these small scale features to initial conditions in the physical domain can then be traced to the problem of determination of the initial singularity location in the complex plane from finite precision data in the real domain. Furthermore, from (27), we note that the problem of determining the singularity trajectory is the complex equivalent of determining fluid particle trajectories in the real domain, once an Eulerian flow field is determined. In such cases, "Langrangian chaos" is a well-known phenomena. Thus, we speculate that the theory of dynamical systems may be relevant to describing chaotic complex singularity trajectories, which could have a bearing on turbulence.

As far as the hydrodynamic stability of steady flows is concerned, we have introduced two new conjectures that connect the complex singularity motion with the growth of a disturbance in the real domain. Admittedly, more theoretical work is necessary to put these conjectures on a firm foundation. Nonetheless, the application of Conjecture 1 to specific examples that include the class of elliptical flows and the Kelvin-Stuart Cat's Eye give results in agreement with previously determined results. Though no new instabilities have been discovered so far, the ease and simplicity with which these results have been found is certainly encouraging. Indeed, as mentioned before, Conjecture 1 can even be be applied to a numerically determined smooth steady Euler flow provided there is at least one stagnation point where the velocity gradient tensor has an eigenvalue with a negative real part.

So far, we have only considered the stability of Euler flows with no boundaries. We do not expect boundaries to alter the statement of Conjecture 1 – provided that the stagnation point is in the interior of the domain – since the instability mechanism associated with the approach of a complex singularity is associated with small scale disturbances in the real domain. This issue will be the subject of a future study.

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APPENDIX

In Section 3, it has been shown that if the velocity gradient tensor $\mathbf{T}$ at a stagnation point $\vec{x}_s$ (taken to be zero without any loss of generality) has an eigenvalue σ with a negative real part, then there exists a singularity trajectory approaching the stagnation point such that, for sufficiently large t, the singularity trajectory $\vec{x}_0(t)$ asymptotes to

$$\vec{x}_0(t) = \vec{x}_0(t_0)e^{\sigma(t-t_0)} \tag{A1}$$

In that case, it was shown that $\vec{p}(\vec{x}_0(t_0), t_0)$ can be chosen such that for sufficiently large t, $\vec{p}(\vec{x}_0(t), t)$ asymptotes to

$$\vec{p}(\vec{x}_0(t), t) = e^{\lambda(t-t_0)}\vec{p}(\vec{x}_0(t_0), t_0) \tag{A2}$$

where λ is an eigenvalue of $\mathbf{T}$ with positive real part.

The purpose of this Appendix is to show that the exponential growth of $\vec{p}$ in time (as in (A2)) at the moving singular point $\vec{x}_0(t)$ that approaches the origin (i.e. the stagnation point, as in (A1)) corresponds to a growth of the disturbance vorticity $\vec{\omega}_D = f'(d)\vec{p}$ at the origin. Here, we only do so for the case when the the velocity gradient tensor at every point is $\mathbf{T}$, a constant. We further assume that the disturbance is small enough so that the following linearized equations hold

$$\frac{\partial \vec{\omega}_D}{\partial t} + \vec{u}_s \cdot \nabla \vec{\omega}_D + \vec{u}_D \cdot \nabla \vec{\omega}_s = \vec{\omega}_D \cdot \nabla \vec{u}_s + \vec{\omega}_s \cdot \nabla \vec{u}_D \tag{A3}$$

with

$$\nabla \times \vec{u}_D = \vec{\omega}_D \tag{A4}$$

$$\nabla \cdot \vec{u}_D = 0 \tag{A5}$$

where $\vec{u}_D = f(d)\,\vec{q}$.

We introduce new variables,

$$\vec{x}e^{-\sigma(t-t_0)} = \vec{y} \tag{A6}$$

$$e^{\sigma(t-t_0)}\vec{\omega}_D = \vec{W} \tag{A7}$$

Then, with the assumption of constancy of $\nabla \vec{u}_s = \mathbf{T}$ (and therefore of $\vec{\omega}_s$), we have

$$\vec{W}_t - \sigma \vec{W} - \sigma \vec{Y} \cdot \nabla_Y \vec{W} + \mathbf{T}\vec{Y} \cdot \nabla_Y \vec{W} = \mathbf{T}\vec{W} + \vec{\omega}_s \cdot \nabla_Y \vec{u}_D \tag{A8}$$

$$\nabla_Y \times \vec{u}_D = \vec{W} \tag{A9}$$

$$\nabla_Y \cdot \vec{u}_D = 0 \tag{A10}$$

where the subscript Y denotes a derivative with respect to the $\vec{Y}$ components.

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Equations (A8) - (A10) are linear equations with no explicit time dependence. Assuming that the eigenvectors corresponding to the linear stability operator form a complete set, the large t behavior of the solution for a generic initial condition can be expected to be dominated by the least stable (or most unstable) eigenvector times $e^{\delta t}$ (multiplied at best by a polynomial in t) where δ is the eigenvalue with the largest real part. The analytical continuation of such a solution to the complex plane will have similar time dependence. Thus, according to this argument, the time dependence of $\vec{W}$ at a fixed complex $\vec{Y} = \alpha \vec{x}_0(t_0)$ can only be of the form $e^{\delta t}$ (multiplied at best by a polynomial in t).

However, for the specific initial condition discussed in Section 3, that leads to (A2), it follows from the transformations (A6) and (A7) that the time dependence of $\vec{W}(\alpha \vec{x}_0(t_0), t)$ is of the form $e^{(\lambda+\sigma)t}$. This growth rate (or decay rate if $Re\ (\lambda + \sigma)\ <\ 0$)) cannot exceed (or have a decay rate less than) $e^{\delta t}$ times a polynomial in t. Thus, $Re\ \delta\ \geq\ Re(\ \lambda + \sigma)$ implying that for a generic non-zero choice of $\vec{W}(0, t_0)$, the growth rate of $\vec{W}(0, t_0)$ will equal or exceed $e^{Re(\lambda+\sigma)t}$. From transformation (A7), it follows that $\vec{\omega}_D(0, t)$ will then grow at a rate at least equaling $e^{Re\lambda t}$, implying absolute instability.

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Equations governing the motion of a specific class of singularities of the Euler equation in the extended complex spatial domain are derived. Under some assumptions, it is shown how this motion is dictated by the smooth part of the complex velocity at a singular point in the unphysical domain. These results are used to relate the motion of complex singularities to the stability of steady solutions of the Euler equation. A sufficient condition for instability is conjectured. Several examples are presented to demonstrated the efficacy of this sufficient condition which include the class of elliptical flows and the Kelvin-Stuart Cat's Eye.

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